FUNCTIONAL EQUATIONS FOR ZETA FUNCTIONS OF $\mathbb{F}_1$-SCHEMES

OLIVER LORSCHEID

ABSTRACT. For a scheme X whose $\mathbb{F}_q$-rational points are counted by a polynomial $N(q) = \sum a_i q^i$, the $\mathbb{F}_1$-zeta function is defined as $\zeta_{\mathcal{X}}(s) = \prod (s-i)^{-a_i}$. Define $\chi = N(1)$. In this paper we show that if X is a smooth projective scheme, then its $\mathbb{F}_1$-zeta function satisfies the functional equation $\zeta_{\mathcal{X}}(n-s) = (-1)^{\chi}\zeta_{\mathcal{X}}(s)$. We further show that the $\mathbb{F}_1$-zeta function $\zeta_{\mathcal{G}}(s)$ of a split reductive group scheme G of rank r with N positive roots satisfies the functional equation $\zeta_{\mathcal{G}}(r+N-s) = (-1)^{\chi}\big(\zeta_{\mathcal{G}}(s)\big)^{(-1)^r}$.

1. INTRODUCTION

In recent years around a dozen different suggestion of what a scheme over $\mathbb{F}_1$ should be appeared in literature (cf. [6]). The common motivation for all these approaches is to provide a framework in which Deligne's proof of the Weyl conjectures can be transfered to characteristic 0 in order to proof the Riemann hypothesis. Roughly speaking, $\mathbb{F}_1$ should be thought of as a field of coefficients for $\mathbb{Z}$, and $\mathbb{F}_1$-schemes $\mathcal{X}$ should have a base extension $\mathcal{X}_{\mathbb{Z}}$ to $\mathbb{Z}$ which is a scheme in the usual sense.

Though it is not clear yet whether one of the existing $\mathbb{F}_1$-geometries comes close to this goal, and thus in particular it is not clear what the appropriate notion of an $\mathbb{F}_1$-scheme should be, the zeta function $\zeta_{\mathcal{X}}(s)$ of such an elusive $\mathbb{F}_1$-scheme $\mathcal{X}$ is determined by the scheme $X = \mathcal{X}_{\mathbb{Z}}$.

Namely, let X be a variety of dimension n over $\mathbb{Z}$, i.e. a scheme such that X_k is an variety of dimension n for any field k. Assume further that X has a counting polynomial

$$N(q) \quad = \quad \sum_{i=0}^{n} a_i\, q^i \qquad \in \qquad \mathbb{Z}[q],$$

i.e. the number of $\mathbb{F}_q$-rational points is counted by $\#X(\mathbb{F}_q) = N(q)$ for every prime power q. If X descents to an $\mathbb{F}_1$-scheme $\mathcal{X}$, i.e. $\mathcal{X}_{\mathbb{Z}} \simeq X$, then $\mathcal{X}$ has the zeta function

$$\zeta_{\mathcal{X}}(s) \quad = \quad \lim_{q\to 1} \quad (q-1)^{\chi}\, \zeta_X(q,s)$$

where $\zeta_X(q,s) = \exp\left(\sum_{r\geq 1} N(q^r) q^{-sr}/r\right)$ is the zeta function of $X \otimes \mathbb{F}_q$ if q is a prime power and $\chi = N(1)$ is the order the pole of $\zeta_X(q,s)$ in $q=1$ (cf. [9]). This expression comes down to

$$\zeta_{\mathcal{X}}(s) \quad = \quad \prod_{i=0}^{n} (s-i)^{-a_i}$$

([9, Lemme 1]).

From this it is clear that $\zeta_{\mathcal{X}}(s)$ is a rational function in s and that its zeros (resp. poles) are at $s=i$ of order $-a_i$ for $i = 0,\dots,n$. The only statement from the Weyl conjectures which is not obvious for zeta functions of $\mathbb{F}_1$-schemes is the functional equation.

I like to thank Takashi Ono for drawing my attention to the symmetries occuring in the counting polynomials of split reductive group schemes, and I like to thank Markus Reineke for his explanations on the comparision theorem for liftable smooth varieties.

2. The functional equation for smooth projective $\mathbb{F}_1$-schemes

Let X be an (irreducible) smooth projective variety of dimesion n with a counting polynomial $N(q)$. Let $b_0, \dots, b_{2n}$ be the Betti numbers of X, i.e. the dimensions of the singular homology groups $H_0(X_\mathbb{C}), \dots, H_{2n}(X_\mathbb{C})$. By Poincaré duality, we know that $b_{2n-i} = b_i$. As a consequence of the comparision theorem for smooth liftable varieties and Deligne's proof of the Weil conjectures, we know that the counting polynomial is of the form

$$N(q) \quad = \quad \sum_{i=0}^{n} b_{2i}\, q^i$$

and that $b_i = 0$ if i is odd (cf. [2] and [8]). Thus $\chi = \sum_{i=0}^{n} b_{2i}$ is the Euler characteristic of $X_\mathbb{C}$ in this case (cf. [4]).

Suppose X has an elusive model $\mathcal{X}$ over $\mathbb{F}_1$. Then $\mathcal{X}$ has the zeta function $\zeta_\mathcal{X}(s) = \prod_{i=0}^{n} (s-i)^{-b_{2i}}$.

Theorem 1. *The zeta function $\zeta_\mathcal{X}(s)$ satisfies the functional equation*

$$\zeta_\mathcal{X}(n-s) \quad = \quad (-1)^\chi\, \zeta_\mathcal{X}(s)$$

and the factor equals -1 if and only if n is even and b_n is odd.

Proof. We calculate

$$\begin{aligned} \zeta_\mathcal{X}(n-s) \quad &= \quad \prod_{i=0}^{n} ((n-s)-i)^{-b_{2i}} \\ &= \quad \prod_{i=0}^{n} (-1)^{b_{2i}} (s-(n-i))^{-b_{2i}} \\ &= \quad (-1)^\chi \prod_{i=0}^{d} (s-(n-i))^{-b_{2n-2i}} \end{aligned}$$

where we used $b_{2n-2i} = b_{2i}$ in the last equation. If we now substitute i by $n-i$ in this expression, we obtain

$$\zeta_\mathcal{X}(n-s) \quad = \quad (-1)^\chi \prod_{i=0}^{n} (s-i)^{-b_{2i}} \quad = \quad (-1)^\chi\, \zeta_\mathcal{X}(s).$$

If n is odd, then there is an even number of non-trivial Betti numbers and $\chi = 2b_0 + 2b_2 + \cdots + 2b_{n-1}$ is even. If n is odd, then $\chi = 2b_0 + 2b_2 + \cdots + 2b_{n-2} + b_n$ has the same parity as b_n. Thus the additional statement. $\square$

Remark. Note the similarity with the functional equation for motivic zeta functions as in [3, Thm. 1]. Amongst other factors, also $(-1)^{\chi(M)}$ appears in the functional equation of the zeta function of a motive M where $\chi(M)$ is the (positive part of the) Euler characteristic of M.

3. The functional equation for reductive groups over $\mathbb{F}_1$

The above observations imply further a functional equation for reductive group schemes over $\mathbb{F}_1$. Note that Soule's and Connes and Consani's approaches towards $\mathbb{F}_1$-geometry indeed succeeded in descending split reductive group schemes from $\mathbb{Z}$ to $\mathbb{F}_1$ (cf. [1], [5], [7]).

Let G be a split reductive group scheme of rank r with Borel group B and maximal split torus $T \subset B$. Let N be the normalizer of T in G and $W = N(\mathbb{Z})/T(\mathbb{Z})$ be the Weyl group. The Bruhat decomposition of G (with respect to T and B) is the morphism

$$\coprod_{w \in W} BwB \quad \longrightarrow \quad G\ ,$$

induced by the subscheme inclusions $BwB \to G$, which has the property that it induces a bijection between the k-rational points for every field k. We have $B \simeq \mathbb{G}_m^r \times \mathbb{A}^N$ as schemes where N is the number of positive roots of G, and $BwB \simeq \mathbb{G}_m^r \times \mathbb{A}^{N+\lambda(w)}$ where $\lambda(w)$ is the length of $w \in W$. With this we can calculate the counting polynomial of G as

$$N(q) \quad = \quad \# \coprod_{w \in W} BwB(\mathbb{F}_q) \quad = \quad (q-1)^r q^N \sum_{w \in W} q^{\lambda(w)}.$$

The quotient variety G/B is a smooth projective scheme of dimension N with counting function $N_{G/B}(q) = \big((q-1)^r q^N\big)^{-1} N(q) = \sum_{w \in W} q^{\lambda(w)}$. Let $b_0, \dots, b_{2N}$ be the Betti numbers of G/B, then we know from the previous section that $N_{G/B}(q) = \sum_{l=0}^{N} b_{2l} q^l$ and that $b_{2N-2l} = b_{2l}$.

Thus we obtain for the counting polynomial of G that

$$\begin{aligned} N(q) \quad &= \quad q^N \ \left(\sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} q^k \right) \ \cdot \ \left(\sum_{l=0}^{N} b_{2l} q^l \right) \\ &= \quad \sum_{i=0}^{d} \ \left(\sum_{k+l=i-N} (-1)^{r-k} \binom{r}{k} b_{2l} \right) \ q^i \end{aligned}$$

where $d = r + 2N$ is the dimension of G and with the convention that $\binom{r}{k} = 0$ if $k < 0$ or $k > r$. Denote by $a_i = \sum_{k+l=i-N} (-1)^{r-k} \binom{r}{k} b_{2l}$ the coeffients of $N(q)$.

Lemma. *We have* $a_0 = \cdots = a_{N-1} = 0$ *and* $a_{d-i} \ = \ (-1)^r \ a_{i+N}$.

Proof. The first statement follows from the fact that $N(q)$ is divisible by q^N. For the second statement we use the symmetries $\binom{r}{k} = \binom{r}{r-k}$ and $b_{2N-2l} = b_{2l}$ to calculate

$$\begin{aligned} a_{d-i} \quad &= \quad \sum_{k+l=d-i-N} (-1)^{r-k} \binom{r}{k} b_{2l} \\ &= \quad \sum_{k+l=d-i-N} (-1)^r (-1)^k \binom{r}{r-k} b_{2N-2l}. \end{aligned}$$

When we substitute k by $r-k$ and l by $N-l$ in this equation and use $d = r + 2N$, then we obtain

$$a_{d-i} \quad = \quad (-1)^r \sum_{k+l=(i+N)-N} (-1)^{r-k} \binom{r}{k} b_{2l},$$

which is the same as $(-1)^r a_{i+N}$. □

Suppose G has an elusive model $\mathcal{G}$ over $\mathbb{F}_1$. Then $\mathcal{G}$ has the zeta function $\zeta_{\mathcal{G}}(s) = \prod_{i=0}^{n} (s-i)^{-a_i}$. Let $\chi = N(1) = \sum_{i=0}^{d} a_i$.

Theorem 2. *The zeta function* $\zeta_{\mathcal{G}}(s)$ *satisfies the functional equation*

$$\zeta_{\mathcal{G}}(r+N-s) \quad = \quad (-1)^{\chi} \left(\zeta_{\mathcal{G}}(s) \right)^{(-1)^r}.$$

Proof. We use of the previous lemma and $r+N=d-N$ to calculate that

$$\begin{aligned}\zeta_{\mathcal{G}}(r+N-s) &= \prod_{i=0}^{n}(r+N-s-i)^{-a_i} \\ &= \prod_{i=0}^{n}(d-N-s-i)^{-(-1)^r a_{d-N-i}}.\end{aligned}$$

After substituting i by $d-N-i$, we find that

$$\begin{aligned}\zeta_{\mathcal{G}}(d-N-s) &= \prod_{i=0}^{n}(i-s)^{-(-1)^r a_i} \\ &= (-1)^{\sum a_i}\Big(\prod_{i=0}^{n}(s-i)^{-a_i}\Big)^{(-1)^r} \\ &= (-1)^{\chi}\left(\zeta_{\mathcal{G}}(s)\right)^{(-1)^r}.\end{aligned}$$

□

Remark. Kurokawa calculates the $\mathbb{F}_1$-zeta functions of $\mathbb{P}^n$, $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ in [4]. One can verify the functional equation for these examples immediately.

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CITY COLLEGE OF CUNY, DEPT. MATH., 160 CONVENT AVE., NYC, NY 10031, U.S.
E-mail address: `olorscheid@ccny.cuny.edu`